Short Communication

Self and mutual radiation from flexural and radial modes of a thick annular disk

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Abstract

This article introduces an analytical modal expansion technique for examining the acoustic radiation coupling between interacting structural modes. This procedure is applied to a thick annular disk (with free boundaries) to investigate self and mutual radiation from in-plane (q) and out-of-plane (m, n) modes. Also, the sound power generated by an arbitrary harmonic force is calculated from self and mutual radiation powers. According to the results of analytical and numerical investigations, sound powers due to modal coupling exist only when two modes have same n or q or when $n = q$. Finally, the sound power and radiation efficiency spectra, under the application of two simultaneously applied harmonic forces, are calculated by using the expansion procedure.
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1. Introduction

Sound radiation from the multimodal flexural vibrations of thin beam or plate structures has been investigated and some modal coupling effects have been examined [1–4]. In more complex and thick structures such as brake rotors, gear blanks, flywheels, or machine casings, one would expect that both in-plane and out-of-plane (flexural) modes would radiate sound while creating some interactions among the relevant modes. We examine this concept in the context of a thick

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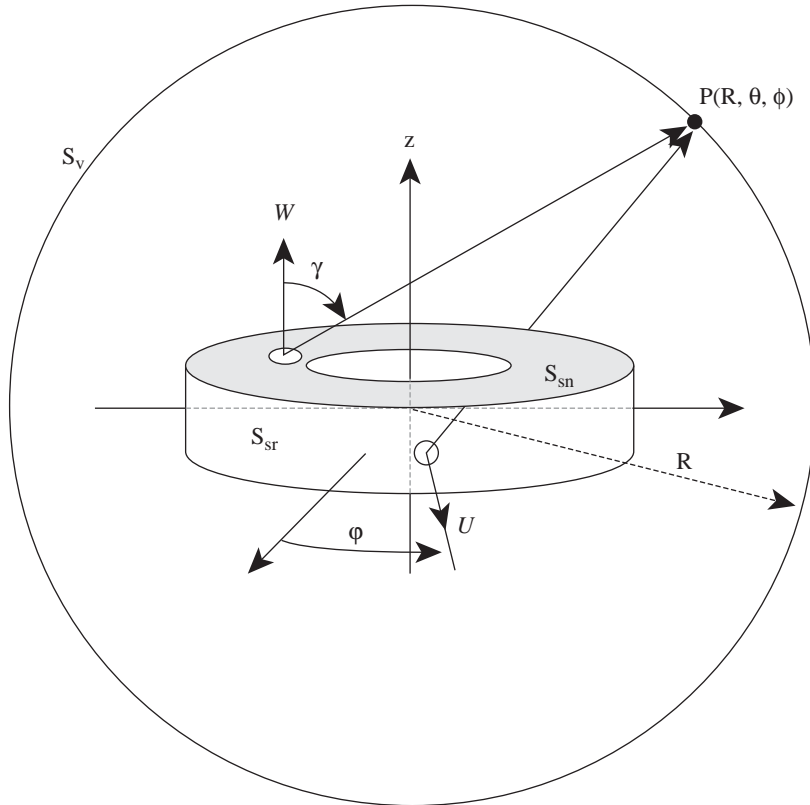*E-mail address:* singh.3@osu.edu (R. Singh).

Fig. 1. Sound radiation from a vibrating thick annular disk with free boundaries. Both flexural and radial modes radiate sound.

annular disk with radial modes (index q) and flexural modes (with m nodal circles and n nodal diameters). Sound radiation from this disk with free–free boundaries is conceptually shown in Fig. 1 in terms of source and receiver positions. The scope of this study is strictly limited to the frequency domain analysis and complicating effects such as fluid loading and scattering at the disk surfaces are not considered.

In a recent paper, Lee and Singh [5] proposed an analytical solution for sound radiation from the (m, n)th flexural modes of a thick annular disk that specifically considers the disk thickness (h) effect. With reference to Fig. 2(a), the modal sound pressure is expressed by a sum of sound radiations from two normal surfaces (at $z = 0.5$ and $-0.5\,\text{h}$) as

$$P_{m,n}(R, \theta, \phi) = (1 + \cos\theta)P_{m,n}^s(R, \theta, \phi) + (1 - \cos\theta)P_{m,n}^o(R, \theta, \phi),$$

$$P_{m,n}^s(R, \theta, \phi) = \frac{\rho_0 c_0 k_{m,n} e^{ik_{m,n}R}}{2R} e^{-ik_{m,n}(h/2)\cos\theta} \cos n\phi(-i)^{n+1} B_n\big[\dot{W}(r)\big],$$

$$P_{m,n}^o(R, \theta, \phi) = \frac{\rho_0 c_0 k_{m,n} e^{ik_{m,n}R}}{2R} e^{ik_{m,n}(h/2)\cos\theta} \cos n(\phi + \pi)(-i)^{n+1} B_n\big[\dot{W}(r)\big]. \qquad (1a\text{–}c)$$
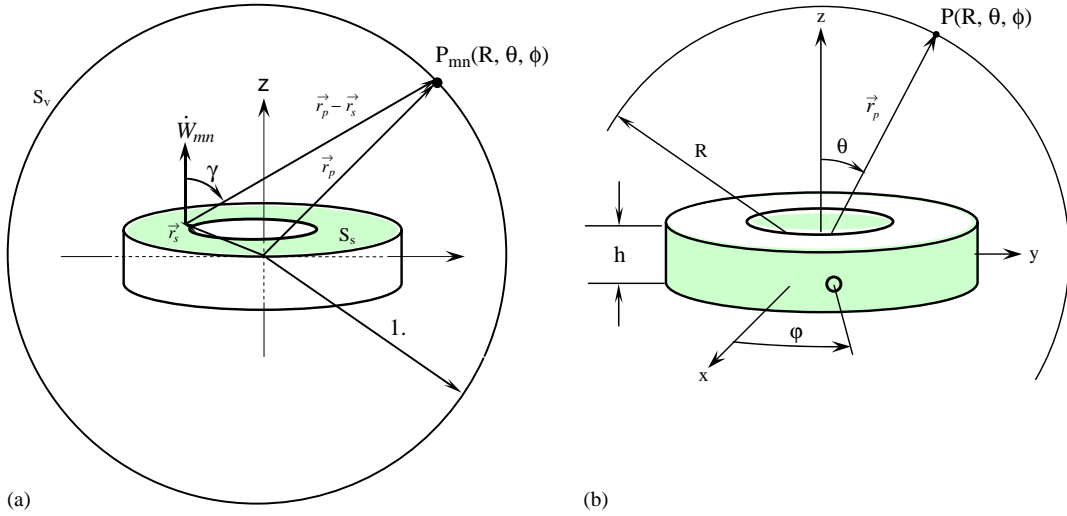
Fig. 2. Sound radiation from a thick annular disk in the spherical coordinate system: (a) from the out-of-plane (flexural) structural modes; (b) from the radial structural modes.

Here, ρ_0 is the mass density of air, c_0 is the speed of sound, $k_{m,n}$ is the acoustic wavenumber of the (m,n)th mode, $\dot{W}(r)$ is surface velocity variation in radial direction and $B_n\left[\dot{W}(r)\right] = \int_0^\infty \dot{W}(r)J_n(k_r r)r\,\mathrm{d}r$. Using Eq. (1), the far-field P values at predetermined observation points on S_V (centered at the disk origin) are calculated. In addition, the modal sound radiation ($\Gamma_{m,n}$) is determined from the sound pressure distribution on S_V. Details of this analytical solution along with numerical and experimental verification are available in the previous article [5].

Far-field sound pressures from the radial (q) modes of a thick annular disk are calculated by using the cylindrical radiator approach [6]. The total sound pressure P_q at the receiver position is expressed as a sum of P_{qO} from the outer radial surface and P_{qI} from the inner radial surface. Refer to Fig. 2(b) for the relevant configuration in the spherical coordinate system:

$$P_q(R, \theta, \phi) = P_{qI}(R, \theta, \phi) + P_{qO}(R, \theta, \phi),$$

$$P_{qO}(R, \theta, \phi) = \frac{\rho_0 e^{ik_q R}}{\pi k_q R \sin\theta} |\ddot{u}_{qO}| h \frac{\mathrm{Sinc}(k_q \sin\theta h/2)(-i)^{q+1}}{H_q^{1'}(k_q a \sin\theta)} \cos q\phi,$$

$$P_{qI}(R, \theta, \phi) = \frac{\rho_0 e^{ik_q R}}{\pi k_q R \sin\theta} |\ddot{u}_{qI}| h \frac{\mathrm{Sinc}(k_q \sin\theta h/2)(-i)^{q+1}}{H_q^{2'}(k_q b \sin\theta)} \cos q\phi,$$

$$\mathrm{Sinc}(x) = \frac{\sin(x)}{x}. \tag{2a--d}$$

Here, k_q is the acoustic wavenumber of the qth mode, H_q is the qth-order Hankel function, $|\ddot{u}_{qO}|$ and $|\ddot{u}_{qI}|$ are the amplitudes of vibratory acceleration on outer and inner radial edges, respectively, and a and b are outer and inner radius of the disk. As in the case of out-of-plane mode, the modal sound radiation (Γ_q) are defined from calculated sound pressures on S_v. Details and verification of this method can be found in Ref. [6].

Chief objectives of this communication are as follows, given the fundamental modal radiation solutions as described above: (1) examine the interactions between in-plane and out-of-plane modes, as well as couplings within the same type of structural modes, from the sound radiation perspective; (2) define the acoustic powers (Π) from self and mutual radiations of a thick annular disk; (3) calculate the sound pressure amplitudes (P) due to a multidirectional harmonic force and investigate the effect of two simultaneous forces (when separated by a circumferential distance) on sound powers.

2. Multimodal vibro-acoustic responses to an arbitrary harmonic force

The multimodal vibro-acoustic responses of the disk to a harmonic force when applied in an arbitrary direction and at frequency ω can be obtained by the modal expansion technique [7]:

$$\{v(\omega)\} = \{\Phi^V\}\{\eta(\omega)\}^{\mathrm{T}}, P(\omega) = \{\Gamma\}\{\eta(\omega)\}^{\mathrm{T}},$$

$$\{\Phi^V\} = \left\{ \Phi^V_{0,2,-1}, \Phi^V_{1,0,-1}, \Phi^V_{0,3,-1}, \ldots, \Phi^V_{m,n,-1}, \Phi^V_{-1,-1,2}, \Phi^V_{-1,-1,0}, \ldots, \Phi^V_{-1,-1,q} \right\},$$

$$\{\Gamma\} = \{\Gamma_{0,2,-1}, \Gamma_{1,0,-1}, \Gamma_{0,3,-1}, \ldots, \Gamma_{m,n,-1}, \Gamma_{-1,-1,2}, \Gamma_{-1,-1,0}, \ldots, \Gamma_{-1,-1,q}\}, \quad (3a\text{–}e)$$

$$\{\eta(\omega)\} = \left\{ \eta_{0,2,-1}(\omega), \eta_{1,0,-1}(\omega), \ldots, \eta_{m,n,-1}(\omega), \eta_{-1,-1,2}(\omega), \ldots, \eta_{-1,-1,q}(\omega) \right\},$$

where $v(\omega)$ is the surface velocity distribution in the disk, $P(\omega)$ is the far-field sound pressure on the sphere S_V, $\Phi^V_{m,n,q}$ is the velocity modal vector of the disk that is expressed as the displacement modal vector multiplied by corresponding natural frequency, $\eta_{m,n,q}(\omega)$ are the modal participation factors for the given force excitation and $\Gamma_{m,n,q}$ is the modal sound radiation solution as obtained from Eqs. (1) and (2). In Eq. (3), a new modal index notation (m, n, q) has been introduced that simultaneously represents out-of-plane and radial modes. The value of -1 for any index is used to represent a null; that would imply that the corresponding modal characteristics would not occur in the newly defined modal property. For instance, $\Phi^V_{0,2,-1}$ is the velocity mode shape of a $(0, 2)$ mode and $\Gamma_{-1,-1,2}$ is the modal sound radiation of a pure $q = 2$ mode. Modal participation factors at frequency ω due to a multidirectional harmonic force $f(t)$ applied at (r_f, φ_f) are obtained from the modal data as

$$\eta_{m,n,q}(\omega) = \sum \frac{\Phi_{m,n,q}(r_f, \varphi_f)\Phi^{\mathrm{T}}_{m,n,q}(r, \varphi)}{\left\{ 1 - \omega^2/\omega^2_{m,n,q} \right\} + i\left\{ 2\varsigma_{m,n,q}(\omega/\omega_{m,n,q}) \right\}}. \quad (4)$$

Here, $\omega_{m,n,q}$ and $\zeta_{m,n,q}$ are the natural frequency and modal damping ratio of mode (m, n, q), respectively. Corresponding sound power (Π) of the disk and corresponding radiation efficiency (σ) are also calculated from the far-field P values, at frequency ω, on a sphere surrounding the disk as

$$\Pi(\omega) = \left\langle I_s(\omega)S \right\rangle_s = \frac{1}{2} \int_0^{2\pi} \int_0^{\pi} \frac{P^H(\omega)P(\omega)}{\rho_0 c_0} R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi,$$

$$\sigma(\omega) = \frac{\Pi(\omega)}{A_s \left\langle |v|^2 \right\rangle_{t,s}},$$

$$\left\langle |v|^2 \right\rangle_{t,s} = \frac{1}{4\pi h(a+b) + 2\pi(a^2 - b^2)} \left\{ \int_{-h/2}^{h/2} \int_0^{2\pi(a+b)} \dot{U}^2 \, \mathrm{d}l\mathrm{d}z + \int_b^a \int_0^{2\pi} \dot{W}^2 \, \mathrm{d}\varphi \, \mathrm{d}r \right\}. \quad (5a\text{–}c)$$

A detailed description of this procedure can be found in our previous article [7] along with experimental and numerical verification for a sample disk (outer radius $= 151.5$ mm, inner radius $= 82.5$ mm, and $h = 31.5$ mm) with free boundaries.

3. Sound powers from self and mutual radiation terms

If several structural modes are simultaneously excited, the total sound power Π of Eq. (5) at frequency ω can be decomposed into two groups: (1) Π from the self-radiation of individual modes, and (2) Π from the mutual radiation between any two (or more) structural modes. For the case of a thick annular disk case, Eqs. (3) and (5) suggest that coupling would exist between (m, n) and q modes, as well as between any two (m, n) modes or between any two q modes. Thus, define the sound power generated by the coupling between the ith mode (m_i, n_i, q_i) and jth mode (m_j, n_j, q_j) as

$$\Pi_{m_i n_i r_i}^{m_j n_j r_j} = \frac{R^2}{2\rho_0 c_0} \int_0^{2\pi} \int_0^{\pi/2} P_{m_i n_i r_i} P_{m_j n_j r_j}^* \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi. \tag{6}$$

Here, $\Pi_{m_i n_i r_i}^{m_j n_j r_j}$ is the power from self-radiation when $m_i = m_j, n_i = n_j$ and $q_i = q_j$. Otherwise, $\Pi_{m_i n_i r_i}^{m_j n_j r_j}$ is due to the mutual radiation between (m_i, n_i, q_i) and (m_j, n_j, q_j) modes. By repeating the above procedure, radiated powers associated with individual modes of the disk can be obtained along with sound powers due to the coupling effects between any two structural modes. Typical results are summarized in Table 1. The coupling between any two out-of-plane modes exists only when two modes have the same nodal diameters ($n_i = n_j$). Coupled acoustic powers between two in-plane modes of different q numbers are negligible. Finally, coupled sound powers between in-plane and out-of-plane modes exist only when $n = q$; otherwise, the coupling effect is negligible. For example, the (0, 2) mode couples well with (1, 2) and $q = 2$ modes. Further, the total acoustic

Table 1
Self and mutual radiation terms of sound powers between selected structural modes of a thick annular disk with free boundaries

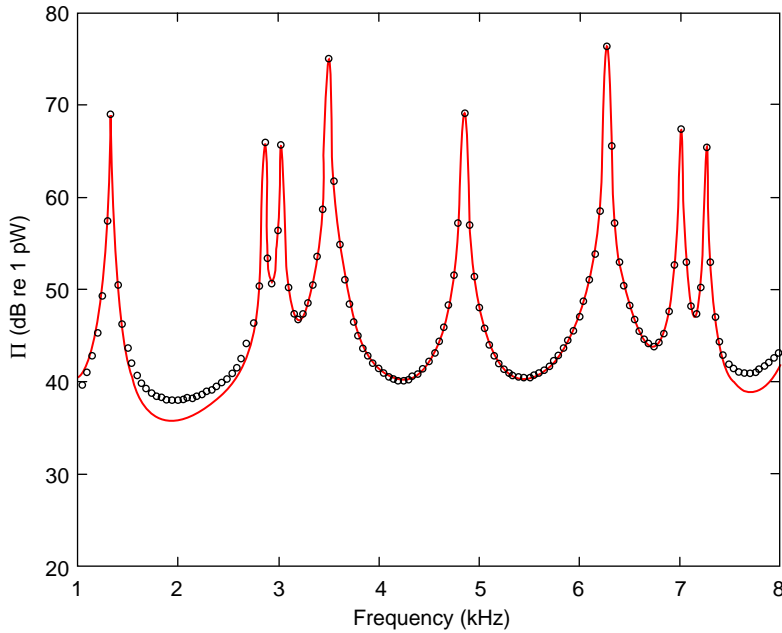| m, n, q		First interacting mode								
		0,2,−1	1,0,−1	0,3,−1	1,1,−1	0,4,−1	1,2,−1	−1,−1,2	−1,−1,3	−1,−1,0
Second	0,2,−1	8.9E+6	0	0	0	0	3.8E+6	2.2E+7	0	0
interacting	1,0,−1	0	7.9E+7	0	0	0	0	0	0	2.4E+7
mode	0,3,−1	0	0	2.2E+8	0	0	0	0	3.6E+8	0
	1,1,−1	0	0	0	1.9E+8	0	0	0	0	0
	0,4,−1	0	0	0	0	7.8E+8	0	0	0	0
	1,2,−1	3.8E+6	0	0	0	0	6.8E+8	2.0E+7	0	0
	−1,−1,2	2.2E+7	0	0	0	0	2.0E+7	1.2E+8	0	0
	−1,−1,3	0	0	3.6E+8	0	0	0	0	9.3E+8	0
	−1,−1,0	0	2.4E+7	0	0	0	0	0	0	1.9E+9

The value of −1 for any index is used to represent a null.

Fig. 3. Comparison of sound power spectra $\Pi(\omega)$ obtained by Eqs. (5) and (7). Key: ∘ ∘ ∘, modal expansion method based on Eq. (5);——, calculation based on self and mutual sound powers as given by Eq. (7).

power due to multimodal excitation could be obtained in terms of a linear combination of the sound powers from self and mutual-radiation terms as

$$\Pi(\omega) = \sum \sum \eta_{m_i n_i q_i}(\omega) \eta_{m_j n_j q_j}(\omega) \Pi_{m_i n_i q_i}^{m_j n_j q_j}. \tag{7}$$

The sound powers $\Pi(\omega)$ for the case of a single harmonic force are calculated using Eq. (7) and compared in Fig. 3 with those from Eq. (5). The calculations based on the self and mutual radiation powers match well with the modal expansion technique. Therefore, we may easily calculate acoustic power for any arbitrary force using Eq. (7).

4. Conclusion

In many practical components, several harmonic forces (say with different frequencies) act simultaneously on a structure but at multiple locations. Therefore, we introduce an analytical approach to address this problem. Recall Eq. (4) where the structural modal participation factors depend on the excitation parameters (r_f, φ_f and ω) as well as on the modal data set ($\omega_{m,n,q}$, $\Phi_{m,n,q}$ and $\zeta_{m,n,q}$). For the sake of illustration, consider two harmonic forces (at the same frequency) that are simultaneously applied to the sample disk. The surface velocity and far-field sound pressure at frequency ω are expressed as the linear combinations of the expressions for individual force

excitations. This procedure is explained below for two forces though it may be generalized:

$$\{v\} = \sum_{j=1}^{2} \{\Phi^v\}\{\eta^j(\omega_j)\}^{\mathrm{T}},$$

$$\{P\} = \sum_{j=1}^{2} \{\Gamma\}\{\eta^j(\omega_j)\}^{\mathrm{T}},$$

$$\{\eta^j(\omega_j)\} = \left\{ \eta^j_{0,2,-1}(\omega_j), \eta^j_{1,0,-1}(\omega_j), \ldots, \eta^j_{m,n,-1}(\omega_j), \eta^j_{-1,-1,2}(\omega_j), \ldots, \eta^j_{-1,-1,q}(\omega_j) \right\}. \tag{8}$$

Here, $\{\eta^j(\omega_j)\}$ is the vector of modal participation factors corresponding to the jth force excitation. As an example, radiation from the sample disk due to two harmonic forces of unity amplitudes is investigated using the proposed procedure. Two forces having the same phase are applied at $r = 151.5\,\mathrm{mm}, \varphi = 0$ and at $r = 151.5\,\mathrm{mm}, \varphi = \Delta\varphi$ positions; refer to Fig. 4 for the configuration. Far-field sound pressure spectra $P(\omega)$ at a selected receiver location $(R = 303\,\mathrm{mm}, \phi = 0, \theta = \pi/4)$ are calculated using Eq. (8) for 4 different circumferential
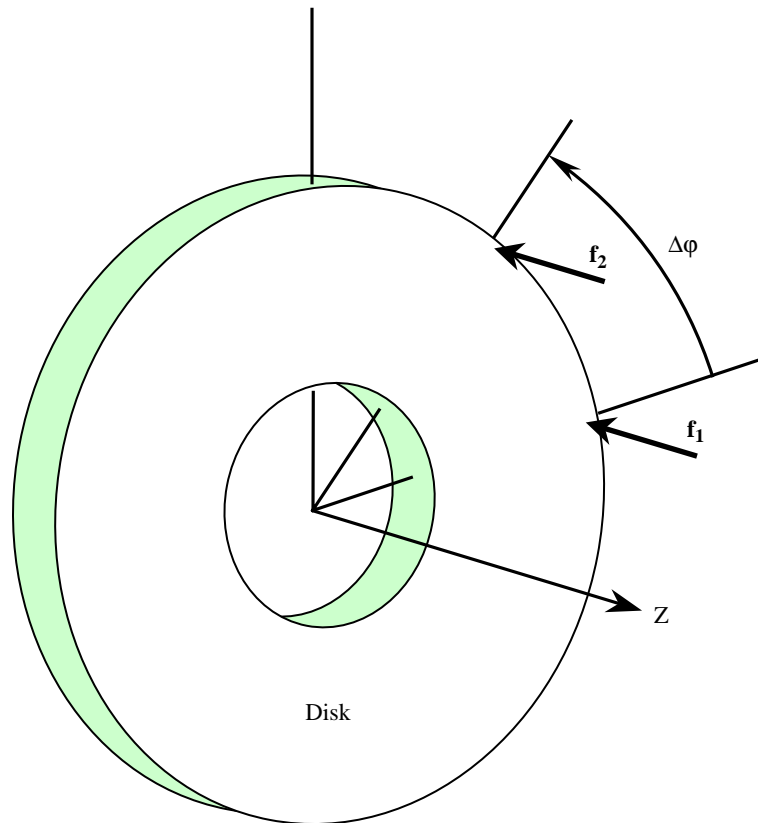

Fig. 4. Calculation of vibro-acoustic responses when excited by two harmonic forces but separated by circumferential distance $\Delta\varphi$.

separation of force locations ($\Delta\varphi = \pi/12, \pi/6, \pi/4$ or $\pi/3$). Results are given in Fig. 5(a). Sound power $\Pi(\omega)$ and radiation efficiency spectra $\sigma(\omega)$ are then calculated using $P(\omega)$ and plotted in Fig. 5(b) and (c), respectively. Observe that when $\Delta\varphi$ is equal to π/j where $j = 1, 2, \ldots$, the structural modes that have j nodal diameters ($n = j$) do not contribute to the combined sound
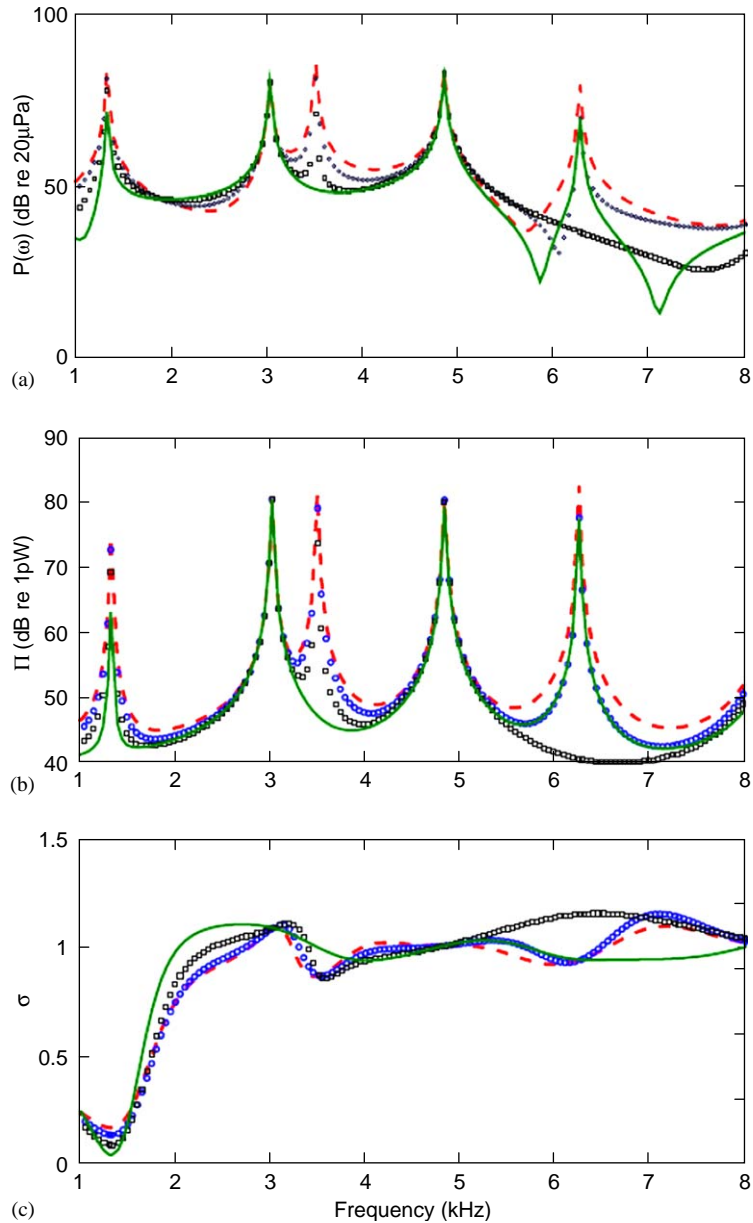

Fig. 5. Sound radiation due to two identical harmonic forces but separated by circumferential distance $\Delta\varphi$ as shown in Fig. 4: (a) Far field $P(\omega)$ at $R = 303$ mm, $\phi = 0$ and $\theta = \pi/4$; (b) sound power spectra $\Pi(\omega)$; (c) radiation efficiency spectra $\sigma(\omega)$. Key: - - -, $\Delta\varphi = \pi/12$; $\circ \circ \circ$, $\Delta\varphi = \pi/6$; $\square \square \square$, $\Delta\varphi = \pi/4$; ——, $\Delta\varphi = \pi/3$.

radiation. Thus, the sound field is significantly affected by the application of two harmonic forces that are circumferentially separated. This suggests that the overall sound radiation could be controlled by passively or actively applying multiple harmonic forces. Finally, future research will extend these concepts further and develop vibro-acoustic design guidelines for brake rotors and gear bodies.

Acknowledgment

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